On Binary Goldbach Conjecture

by

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Summary

I propose in this essay, three problems in analytic number theory, one of them ,if settled , perhaps would be the key to a solution of the **Goldbach** conjecture.

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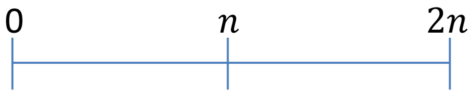
Then corrected many times.

§1) We consider on the real line the interval



where n is a positive integer. According to the PNT the number of primes in $[0, x]$ is given by the asymptotic relation

$$\pi(x) \sim \frac{x}{logx}$$

or similarly the interval $[0, n]$ we have

$$\sim \frac{n}{\log(n)}$$

and in the interval $[0,2n]$ we have

$$\sim \frac{2n}{\log(2n)}$$

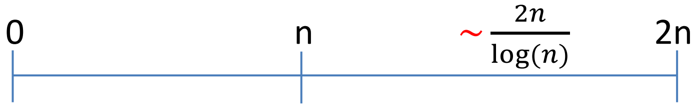
Then in the interval $[n, 2n]$ this becomes, with $\alpha = log2$

$$\sim\ n \times \frac{\log(n) - \alpha}{\log^2(n) + \alpha\log(n)} = \frac{n}{\log(n)} \times \frac{\log(n) - \alpha}{\log(n) + \alpha}$$

or

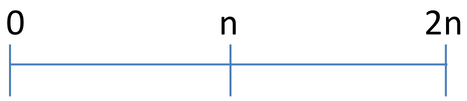
$$\sim 2\,\frac{n}{\log(n)}$$

primes.

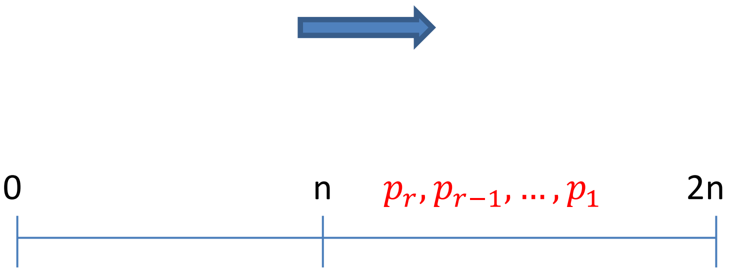


§2) We introduce now the two set of primes: p_i and q_j

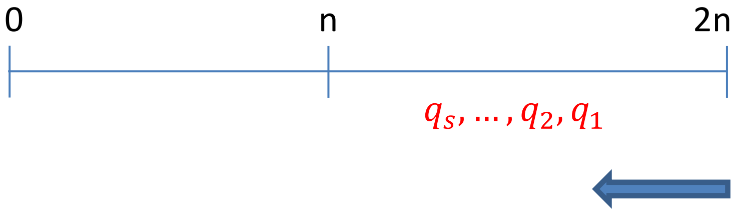
We consider the same interval



We have our two sets of primes, the first



and the second



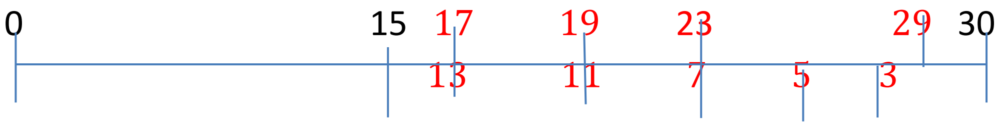
So the pairs (p_i, q_j) encompasses all the primes under $2n$ except 2, such that

$$q_j \leq n \leq p_i < 2n$$

For example, when $n = 15$ we have:

p_i: 17, 19, 23, 29

q_j: 3, 5, 7, 11, 13

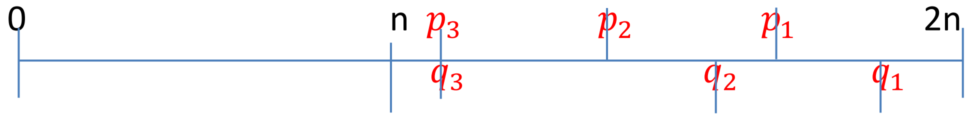


The **Goldbach** conjecture is the hypothesis that says : $\forall n \in \mathbb{N}$

$$2n = p_i + q_j$$

As an example in the following drawing

$$2n = p_3 + q_3$$



By the previous considerations, in the interval $[n, 2n]$ which interests us here , there are

$$\sim \frac{n}{\log(n)}$$

of q_j and

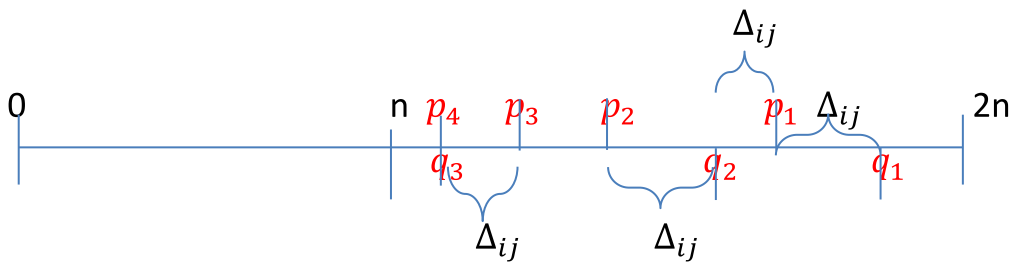
$$\sim n \times \frac{\log(n) - \alpha}{\log^2(n) + \alpha \log(n)}$$

of p_i or

$$\sim 2\,\frac{n}{\log(n)}$$

of p_i and q_j .

§3) Now we consider the *distance* Δ_{ij} between any *two consecutive pairs* of prime numbers (p_i, q_j) :



this number is simply

$$\Delta_{ij} = \left|2n - p_i - q_j\right|$$

where the indices i, j have three possibilities ($i = j$, $i > j$ or $i < j$).

We want to find the *supremum* of Δ_{ij} .

According to the definition of Schnirelmann density (A) , and if we take $a_n = p_i$ (or q_j), we find that

$$d(A) = \inf \quad \left\{2\,\frac{n}{p_i + \Delta_{ij}} \times \frac{1}{logn}\right\} = 0$$

It is evident that $\Delta_{ij} \geq 0$ and it is always even, the case $\Delta_{ij} = 0$ corresponds to the **Goldbach** conjecture, so

$$0 = \min(\Delta_{ij}) = \inf(\Delta_{ij}) < \max(\Delta_{ij}) \leq \sup(\Delta_{ij})$$

Problem 1 :

What is the value of $\sup(\Delta_{ij})$? (i.e. when : $n \longrightarrow +\infty$)

§4) We are interested now in the case when

$$\Delta_{ij} = 2n - p_i - q_j = 0$$

this can be written

$$\Delta_{ij} = n - p_i + n - q_j = 0$$

As we have seen, it corresponds to the **Goldbach** case.

For example , when $n = 17$:

$\Delta = |34 - 3 - 31| = 0$

$\Delta = |34 - 5 - 29| = 0$

$\Delta = |34 - 7 - 23| = 4$

$\Delta = |34 - 11 - 19| = 4$

…

We have remarked through two examples ($n = 15$ and $n = 17$) that the primes p_i and q_i do not go always in pairs, but we will see that it is not necessary for our goal.

So we try to link the **Goldbach** conjecture to Analysis and especially to the theory of *L-functions* : $L(z, \chi)$.

As a very known example, the **Riemann** zeta function ζ has infinitely many roots on the critical line, and infinitely many roots of the form $z = -2n$, otherwise the function $\frac{1}{\zeta}$ has infinitely many singularities on the critical line, and infinitely many singularities of the form $z = -2n$.

We consider the finite sequence for a given n

$$\sum_{2n > p_i \geq n} \sum_{q_j \leq n} \frac{1}{n - p_i + n - q_j}$$

$i = 1,2, \dots, r$, $i = 1,2, \dots, s$, and study the case when the denominator $n - p_i + n - q_j = 0$.

Before that , it is interesting to consider the series

$$\sum_{p \in \mathbb{P}, n \geq 1}^{\infty} \frac{1}{p - n} = \frac{1}{2 - 1} + \frac{1}{3 - 2} + \frac{1}{5 - 3} + \frac{1}{7 - 4} + \frac{1}{11 - 5} + \cdots = -\sum_{p \in \mathbb{P}, n \geq 1}^{\infty} \frac{1}{n - p}$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \cdots$$

And ask:

Problem 2 :

Find an analytic function $\boldsymbol{F}$, such that for certain $L(z,\chi)$ and $\Re(z) > \alpha$ we have:

$$\sum_{p\in\mathbb{P},n\geq 1}^{\infty} \frac{1}{(p-n)^z} = \boldsymbol{F}(L(z,\chi))$$

Similarly, the heart of the **Goldbach** conjecture is the finite sum (or possibly infinite series)

$$\sum_{2n>p_i\geq n} \sum_{q_j\leq n} \frac{1}{(n-p_i)^z + (n-q_j)^z}$$

or

$$\sum_{2n>p_i\geq n} \sum_{q_j\leq n} \frac{1}{(2n-(p_i+q_j))^z}$$

We try to link this finite sum (or series) to some known analytic L-function $L(z,\chi)$, and infer that the **Goldbach** conjecture is true when:

$\underline{\forall n}$, the previous partial sum (or series) has ***at least one singularity***, or the same , verifies the propriety

$$\lim_{z\longrightarrow 1} \quad \sum_{2n>p_i\geq n} \sum_{q_j\leq n} \frac{1}{(n-p_i)^z+(n-q_j)^z} = \infty$$

or $\underline{\forall n}$, the partial sum (or series)

$$\sum_{2n>p_i\geq n} \sum_{q_j\leq n} \frac{1}{(2n-(p_i+q_j))^z}$$

has ***at least one singularity***.

Remark that , from this point of view, we can take in account , in the two cases, the prime $q_j = 2$ too in the partial sum, because :

$$2n-2-p_i \neq 0 \,, \qquad \forall\, n, p_i$$

Then the third problem is

Problem 3 :

Find an analytic function $\boldsymbol{G}$ and $L(z,\chi)$ such that

$$\sum_{2n>p_i\geq n} \sum_{q_j\leq n} \frac{1}{(n-p_i)^z+(n-q_j)^z} = \boldsymbol{G}(L(z,\chi))$$

Through this assay, I think that perhaps the well understanding of the convergent series

$$\sum_{p\in\mathbb{P},n\geq 1}^{\infty}\frac{1}{(p-n)^z}$$

may help us to get the needed information from

$$\sum_{2n>p_i\geq n}\ \sum_{q_j\leq n}\frac{1}{(n-p_i)^z+(n-q_j)^z}$$